Journal of Applied Mechanics and Technical Physics, Vol. 39, No. 2, 1998

EIGENOSCILLATIONS NEAR A PLATE IN A CHANNEL

S. V. Sukhinin

UDC 517.947+534.14+534.2

Acoustic eigenoscillations of a gas near a plate in a rectangular channel, i.e., the eigenfrequency of oscillations as a function of the chord length and the position of the plate in the channel, and the form of the eigenfunctions are studied in a two-dimensional formulation. A mathematical model of eigenoscillations near a plate in a channel has been proposed and substantiated, and the dependence of the eigenfrequency of oscillations on the geometric parameters is studied numerically with the use of this model.

The problem of acoustic oscillations of a gas near a plate in a channel was first formulated in [1–4]. In addition, a mathematical model that describes the self-excited acoustic and electromagnetic oscillations near the obstacle of an arbitrary structure was completely substantiated theoretically, and the existence of eigenfrequencies of oscillations within the framework of the proposed model has been shown. The author proved [4] that the symmetry-breaking of the knife grating does not change its resonance properties.

The presence of the continuous spectrum of frequencies, which corresponds to the generalized eigenfunctions, is the main difficulty in the description of eigenoscillations in unbounded media. This difficulty was overcome in [4] by means of the Fredholm analytical theorem. The existence of eigenoscillations was shown, and the form of these oscillations was examined. Similar propositions obtained by different methods were proved by Evans and Linton [5].

1. FORMULATION OF THE PROBLEM

Equations Describing Acoustic Oscillations. Figure 1 shows the geometry of the acoustic region divided into subregions 1–4. The potential $u(x,y,t)$ of an acoustic velocity perturbation is assumed to be periodically time-dependent: $u(x,y,t) = u(x,y)\exp(i\omega t)$. The equation for the potential of acoustic velocity perturbation $u(\xi,\zeta)$ is of the form

$$u_{\xi\xi} + u_{\zeta\zeta} + \lambda^2 u = 0 \quad \text{in} \quad \Omega, \tag{1.1}$$

where Ω is the region occupied by the gas. The dimensionless frequency λ and variables ξ and ζ are expressed via the dimensional frequency $\lambda = H\omega/c$ and variables $\xi = x/H$ and $\zeta = y/H$, where c is the velocity of sound, H is the height of the channel, and ω is the circumferential frequency of acoustic oscillations.

In dimensionless variables, the channel width equals 1, and to the plate length L corresponds the dimensionless quantity $l = L/H$, which characterizes the length of the plate profile relative to the channel height.

The following Neumann conditions should be satisfied at the channel wall B and the plate profile Γ:

$$u_\zeta = 0 \quad \text{on} \quad \Gamma + B. \tag{1.2}$$

Lavrent'ev Institute of Hydrodynamics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 39, No. 2, pp. 78–90, March–April, 1998. Original article submitted June 10, 1996.

0021-8944/98/3902-0228 $20.00

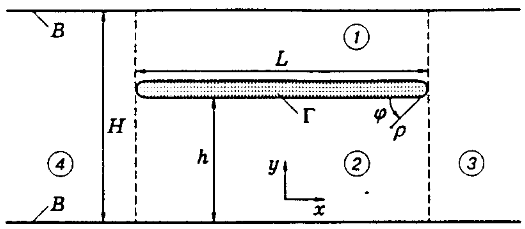

Fig. 1. Geometry of the region of eigenoscillations near the profile in the channel: Γ is the profile of length L, H is the channel height, B are the channel walls, h is the distance from the plate to the lower wall of the channel, (ρ, φ) are the polar coordinates with the origin at the profile edge, and 1–4 refer to the subregions Ω.

According to the physical content of the problem, the condition of energy finiteness should be satisfied in the entire region of oscillations for the function u:

$$E(u) = \int_{\Omega} [u^2 + (\nabla u)^2]\, d\Omega < \infty, \tag{1.3}$$

where $E(u)$ has the sense of energy oscillations.

Radiation Conditions and Continuous Spectrum. It is convenient to choose the coordinate system (ξ, ζ) such that the coordinate origin is at the lower wall of the channel and the ζ axis intersects the plate profile in the center. The channel walls B can be described by the relations $B = \{\zeta = 0, 1\}$, and the profile Γ can be presented as a set on the plane (ξ, ζ): $\Gamma = \{\zeta = h,\ -l/2 \leqslant \xi \leqslant l/2\}$, where h is the distance from the plate to the lower wall of the channel.

Definition 1.1. The solution of Eq. (1.1) is assumed to be subject to the radiation condition if the following representations are true for a rather large number R $(R > L/2)$ and all $\{(\xi, \zeta), |\xi| \geqslant R\}$:

$$\begin{aligned} u(\xi, \zeta) &= \sum_{n=0}^{+\infty} \cos(\pi n \zeta) c_n^{(+)} \exp\left(i\xi\sqrt{\lambda^2 - (\pi n)^2}\right), \quad \xi \geqslant R, \\ u(\xi, \zeta) &= \sum_{n=0}^{+\infty} \cos(\pi n \zeta) c_n^{(-)} \exp\left(-i\xi\sqrt{\lambda^2 - (\pi n)^2}\right) \quad \xi \leqslant -R. \end{aligned} \tag{1.4}$$

Here and below, the dimensionless length of the plate profile l is denoted by L for convenience.

If the function is subject to the radiation condition, it either damps or increases in the general case as the exponent with distance from the coordinate origin (the obstacle). This condition was thoroughly discussed in [1–4].

In addition, it is assumed that, for all $\lambda^2 < (\pi n)^2$, a branch of the square root such that $i\sqrt{\lambda^2 - (\pi n)^2} < 0$ is chosen, and $c_n^{(+)}$ and $c_n^{(-)}$ are assumed to be complex numbers such that the series (1.4) converge. In the class of functions that are subject to the radiation condition, problem (1.1) and (1.2) is the Fredholm problem [2], and it has nontrivial solutions only for a discrete (at a certain Riemann surface) set Λ^* of values of the parameter λ in the Helmholtz equation (1.1). In [1–4], the values of $\lambda^* \in \Lambda^*$ are called quasi-eigenvalues of this problem, and the solutions u_*, which correspond to the values of λ^*, are called quasi-eigenfunctions. They are localized in the neighborhood of the plate and can cause resonance phenomena. In the case where the energy of quasi-eigenoscillations is infinite $[E(u_*) < \infty]$, the function u_* describes the classical eigenoscillations. They are localized in the plate's neighborhood and can cause resonance phenomena. In the case where the energy of quasi-eigenoscillations is infinite, the physical meaning of oscillations is not clear.

For integer n and all λ ($|\lambda| > \pi n$), the functions of the form $Y_n = \cos(\pi n\zeta)\exp\left(-i\xi\sqrt{\lambda^2 - (\pi n)^2}\right)$ describe the generalized eigenwaves with unlimited energy in an empty channel. From the viewpoint of the theory of self-conjugate operators, this means that the corresponding self-conjugate extension of the Laplace operator Δ has a continuous spectrum which occupies the entire nonnegative section of a real straight line. The numbers λ^2 correspond to the generalized eigenvalues of this extension of the operator Δ. To the classical eigenvalues corresponds the operator's purely point spectrum imbedded into a continuous spectrum.

Definition 1.2. Henceforth, the generalized eigenwaves in a channel that are described by the function $Y_0 = \exp(i\lambda\xi)$ are called piston modes.

Remark 1.1. The piston mode is the generalized eigenfunction of a channel with and without a plate. This is due to the fact that the functions describing piston modes do not depend on the variable ζ whose direction is perpendicular to the channel's axis (see Fig. 1). If there is a plate in the channel and it is parallel to the channel walls, the form of the functions that describe the generalized eigenwaves in this structure can be significantly different from the form of the generalized eigenwaves of an empty channel.

Restriction of the Class of Functions. The operator that corresponds to the problem of eigenoscillations near a plate in a channel is of a continuous spectrum coinciding with the positive semi-axis of real numbers, which complicates the study of eigenvalues. The restriction of the space of permissible solutions can shift the lower boundary σ_0 of the continuous spectrum from the coordinate origin. This permits us to employ the known variational methods of determining the eigenvalues in the interval $[0, \sigma_0]$.

In what follows, problem (1.1)–(1.4) is called a problem of eigenoscillations (EO). The space of admissible solutions of this problem is the space of functions with local finite energy in the region Ω/Γ; it is denoted by H_s.

Definition 1.3. The value of the parameter λ for which the nontrivial solution u^* subject to (1.3) exists is called an eigenvalue λ^* of the EO problem. The function u^* is called an eigenfunction of the EO problem.

It is noteworthy that the eigenvalues and functions of the EO problem make it possible to describe completely the acoustic resonance phenomena near a plate in a channel, the eigenvalues are imbedded into a continuous spectrum, and the piston mode $Y_0 = \exp(i\lambda\xi)$ is the generalized eigenfunction of the EO problem.

Two approaches are possible to examine eigenoscillations:

(1) If the EO problem possesses mirror symmetry relative to the middle of a channel, the condition of antisymmetry relative to the middle of the channel $\{(\xi, \zeta), \zeta = 1/2\}$ is added. This condition enables us to exclude the piston mode from the space of admissible solutions of the EO problem. In this approach, the continuous spectrum of the self-conjugate extension of the operator $-\Delta$, which corresponds to the EO problem, equals $[\pi^2, \infty)$. The approach is not substantiated if the position of the plate in the channel is arbitrary.

(2) Another approach was used in [4]. The term that corresponds to the piston mode was shown not to exist in the radiation condition (1.4) for the eigenfunction of the EO problem. The existence of the eigenvalues of the EO problem was proved for an arbitrary position of a long enough profile with the use of the Fredholm analytical theorem. Central to the proof is the circumstance that the piston mode is the generalized eigenfunction of the EO problem.

In these approaches, the restriction of the space of admissible solutions of the EO problem leads to a variation of the continuous spectrum and the appearance of a purely point spectrum of the corresponding operator. The first approach uses the symmetry of the problem, and the second approach uses the fact that piston modes are the generalized eigenfunctions of the EO problem. The latter approach is more general. According to the results of the theory of self-conjugate operators, the eigenfunctions have a zero projection in the corresponding space of functions onto an arbitrary piston mode because it is a generalized eigenfunction. Therefore, if the eigenfunction u^* of the EO problem exists, it should be subject to the necessary condition for all the values of λ:

$$\int_\Omega Y_0 u^*\, d\Omega = \int_\Omega \exp(i\xi\lambda)\, u^*\, d\Omega = 0.$$

The condition will be satisfied for all the values of λ if and only if, for all the values of ξ, the identity

$$\int_0^1 u^*(\xi,\zeta)\,d\zeta \equiv 0 \tag{1.5}$$

holds, which restricts the space H_s of admissible solutions of the problem to the space H_0 ($H_0 \subseteq H_s$), which is a subspace of H_s. With condition (1.5) satisfied for all the values of ξ, the EO problem is called a problem of orthogonal eigenoscillations (OEO). The continuous spectrum which corresponds to the OEO problem is the set $\sigma_1 = [\pi^2, \infty)$ on the real semi-axis. In view of this, the OEO eigenvalues are searched for in the interval $(0, \pi^2)$.

Remark 1.2. The antisymmetry conditions for the eigenfunctions of the variable ζ relative to the plate is a partial case of conditions (1.5).

2. EXISTENCE AND FORM OF EIGENFUNCTIONS

The form of the eigenfunctions far from the plate is described using the radiation and energy-finiteness conditions.

For a deep insight into the mechanics of eigenoscillations and the development of algorithms for numerical analysis, it is necessary to know the form of the eigenfunction in the neighborhood of the profile edges.

Form of the Eigenfunction in the Neighborhood of a Plate. It is important to examine the form of eigenfunctions in the neighborhood of the edges of a plate. The physical prerequisites are as follows:

(a) the energy in the neighborhood of an edge is finite;

(b) the edge does not radiate.

Remark 2.1. These prerequisites are equivalent to each other, and they are a consequence of the finiteness condition for the energy of eigenoscillations (1.3).

In the neighborhood of the profile edge, the solution u^* of the EO (or OEO) problem is of the form [6]

$$u^* = \text{const} + \sqrt{\rho}\cos(\varphi/2). \tag{2.1}$$

Here ρ is the distance from the edge in the neighborhood of which the form of the solution is examined to the point (ξ,ζ), and φ is the angle measured from the lower boundary of the profile for the vector $(\xi - L/2, \zeta - h)$ in studying the neighborhood of the leading edge or for the vector $(\xi + L/2, \zeta - h)$ in studying the trailing edge (see Fig. 1).

In the region Ω/Γ, the solution u of the EO (or OEO) problem can be considered smooth enough, and it can, therefore, be represented in the region Ω as

$$u = u_{\mathrm{d}} + u_{\mathrm{c}}, \tag{2.2}$$

where u_{d} is the discontinuous function on the set of points Γ which describes the profile and u_{c} is the continuous function in the entire region Ω.

Proposition 2.1. *Each solution of the EO problem is representable in the form* (2.2). *The function* u_{d} *is continuous at the profile edge, in regions* 3 *and* 4, *and on the right and on the left from the profile* $u_{\mathrm{d}} = 0$ (see Fig. 1), *and it can be written in the form*

$$u_{\mathrm{d}} = \begin{cases} \nu f(\xi), & (\xi,\zeta) \in 1, \\ (\nu - 1) f(\xi), & (\xi,\zeta) \in 2 \end{cases} \tag{2.3}$$

in regions 1 *and* 2 *and above and below the profile.*

In the case where the function u_{d} is subject to the orthogonality condition for a piston mode (1.5) and the coordinates are dimensionless, the equality $\nu = h$ holds.

Proof. By reductio ad absurdum. Let u_+ be the ultimate value of the function u_{d} for $\zeta \to h+0$, and u_- be its ultimate value for $\zeta \to h-0$. Then the function $f(\xi) = u_+ - u_-$ describes the discontinuity intensity at the profile of the function u_{d}, and representation (2.3) holds. If the continuity is violated at the points

t_1 and t_2, i.e., at the profile edges, condition (2.1), which is a consequence of the energy-finiteness condition (1.3), is not satisfied. The expression for the constant ν is derived from the orthogonality condition (1.5). It is worth mentioning that u_d is allowed not to be subject to the energy-finiteness condition. The proposition is proved.

Corollary 2.1. *There is no discontinuity in the velocity potential at the profile edges, and* $f(\xi) = 0$ *for* $\xi = L/2$ *and* $\xi = -L/2$.

We note that, with the coordinate origin chosen properly ($\xi = 0$ is the middle of a profile), the OEO problem converts into itself relative to the replacement of the variables $\xi \to -\xi$. Therefore, any solution u of the problem is representable as follows: $u = u_s + u_a$. Here and below, $u_s(\xi,\zeta) = u_s(-\xi,\zeta)$ and $u_a(\xi,\zeta) = -u_a(-\xi,\zeta)$ are the components of the solution u which are, respectively, symmetrical (even) and antisymmetrical (odd) in ξ. Since the OEO problem is linear, the space H_0 of all admissible solutions can be given as a direct sum of the spaces of solutions which are symmetrical H_s and antisymmetrical H_a in ξ: $H_0 = H_s \oplus H_a$. Owing to this fact and the linear character, the problem is divided into two independent problems for the functions which show even and odd symmetry in ξ; the analysis of these functions is similar, and hence we shall consider the case of ξ-symmetrical solutions. The respective changes for a study of the case of ξ-odd solutions will be indicated below.

Let u_i ($i = 1,\dots,4$) be the restrictions of the solution u of the OEO problem in regions 1–4, respectively. In the class of ξ-even functions, the general solution of the OEO problem in regions 1–3 has the form

$$\begin{gathered} u_1(\xi,\zeta) = a_0\cos(\lambda\xi) + \sum_{m=1}^{\infty} a_m \cos\left[\frac{m\pi(\zeta-h)}{1-h}\right]\cosh\left(\xi\sqrt{\left[\frac{m\pi}{1-h}\right]^2 - \lambda^2}\right), \\ u_2(\xi,\zeta) = b_0\cos(\lambda\xi) + \sum_{m=1}^{\infty} b_m \cos\left[\frac{m\pi(\zeta-h)}{h}\right]\cosh\left(\xi\sqrt{\left[\frac{m\pi}{h}\right]^2 - \lambda^2}\right), \\ u_3(\xi,\zeta) = c_0\exp(i\lambda\xi) + \sum_{k=1}^{\infty} c_k\cos(k\pi\zeta)\exp\left(-\xi\sqrt{(k\pi)^2-\lambda^2}\right). \end{gathered} \tag{2.4}$$

For ξ-odd functions, the representation

$$\begin{gathered} u_1(\xi,\zeta) = a_0\sin(\lambda\xi) + \sum_{m=1}^{\infty} a_m \cos\left[\frac{m\pi(\zeta-h)}{1-h}\right]\sinh\left(\xi\sqrt{\left[\frac{m\pi}{1-h}\right]^2 - \lambda^2}\right), \\ u_2(\xi,\zeta) = b_0\sin(\lambda\xi) + \sum_{m=1}^{\infty} b_m \cos\left[\frac{m\pi(\zeta-h)}{h}\right]\sinh\left(\xi\sqrt{\left[\frac{m\pi}{h}\right]^2 - \lambda^2}\right), \\ u_3(\xi,\zeta) = c_0\exp(i\lambda\xi) + \sum_{k=1}^{\infty} c_k\cos(k\pi\zeta)\exp\left(-\xi\sqrt{(k\pi)^2-\lambda^2}\right) \end{gathered} \tag{2.5}$$

is true.

Conditions (1.5) will be satisfied in the case where

$$c_0 = 0, \quad a_0(1-h) + b_0 h = 0. \tag{2.6}$$

For a function of the form (2.4) [or (2.5)] with conditions (2.6) to be the solution of the OEO problem, the continuity conditions for the solution and its normal derivative, which are called sewing conditions [7] should be satisfied at the boundaries of regions 1–4. Owing to the symmetry of the problem in ξ, it is sufficient that the sewing conditions are satisfied at the boundary of regions 1–3 and 2–3. Let $g_{(1-3)}$ denote the boundary between regions 1 and 3, and $g_{(2-3)}$ denote the boundary between regions 2 and 3. The sewing conditions are of the form

$$u_1 = u_3, \ \frac{\partial u_1}{\partial\xi} = \frac{\partial u_3}{\partial\xi} \text{ on } g_{(1-3)}; \qquad u_2 = u_3, \ \frac{\partial u_2}{\partial\xi} = \frac{\partial u_3}{\partial\xi} \text{ on } g_{(2-3)}. \tag{2.7}$$

Conditions (2.7) imply that the function of the form (2.3) is a weak solution of the OEO problem in the Sobolev energy space. For elliptic equations, the weak solution is known to be automatically a strong solution.

Existence of Eigenoscillations near the Profile of a Plate in a Channel. To prove the correctness of the mathematical description of acoustic eigenoscillations near a profile in a channel, it is necessary to show that eigenoscillations are described by means of the proposed mathematical model at least for some geometrical parameters.

For this purpose, we shall consider auxiliary problems and the "Dirichlet-Neumann bracket" method [8]. Let the Dirichlet conditions (D) $u(\xi,\zeta) = 0$ for $|\xi| = R > L/2$ or the Neumann conditions (N) $u_\xi(\xi,\zeta) = 0$ for $|\xi| = R > L/2$ be satisfied, in addition to the boundary-problem conditions on the sections $G = \{(\xi,\zeta) : \xi = R, 0 \leqslant \zeta \leqslant 1, R > L/2\}$.

For the convenience of further considerations, the OEO problem with the additional condition D will be called an OEO(DR) problem, and this problem with the condition N will be called an OEO(NR) problem. Let λ_{DR}, u_{DR} and λ_{NR}, u_{NR} be the eigenvalues and eigenfunctions of the OEO(DR) and OEO(NR) problems, respectively. Since the condition N extends the space of admissible solutions of the OEO problem and the condition D restricts it, the inequalities that can be derived using the variational formulation of the problem [8] hold for all R ($R > L/2$):

$$\lambda_{\mathrm{NR}} \leqslant \lambda^* \leqslant \lambda_{\mathrm{DR}}. \tag{2.8}$$

Remark 2.2. If for some values of R ($R \geqslant L/2$), the rigorous inequalities $\lambda_{\mathrm{NR}} > 0$ and $0 < \lambda_{\mathrm{DR}} < \pi$ are satisfied, the existence of the eigenvalue of the EO problem follows from relation (2.8).

If $R = L/2$, the condition D is a "soft" radiation condition for oscillations in channels 1 and 2 (see Fig. 1). The dimensionless eigenfrequency of longitudinal oscillations λ_{DL} is calculated by the formula $\lambda_{\mathrm{DL}} = \pi/L$. By virtue of this, to satisfy the inequality $\lambda^* < \pi$, it suffices that the profile length be larger than the channel height ($L > 1$). The rigorous inequality $\lambda_{\mathrm{NR}} > 0$ follows from condition (1.5) for $R > L/2$. This means that the eigenvalue of the Neumann problem for the Laplace operator in the connected region 1 is rigorously larger than zero.

Theorem 2.1 (the sufficient condition for the existence of eigenoscillations near a profile in a channel). *If the dimensionless profile length L satisfies the rigorous inequality $L > 1$, the nontrivial eigenvalues of the* OEO *problem exist.*

Proof. Let $h \to 1$ (or $h \to 0$), h be the ordinate of the position of the profile, and $(h - 1/2)$ be the deflection of the position of the profile from the center of the channel. Then there exists the eigenfunction $u^*(\xi,\zeta)$ of the OEO(DR) problem which is localized in the region $[-L/2, L/2] \times [h, 1]$ and which has the form

$$u^*(\xi,\zeta) = \begin{cases} \cos(\pi\xi/L), & (\xi,\zeta) \in [-L/2, L/2] \times [h, 1], \\ ((h-1)/h)\cos(\pi\xi/L), & (\xi,\zeta) \in [-L/2, L/2] \times [0, h]. \end{cases}$$

To this eigenfunction corresponds the eigenvalue λ^*, which is subject to the relation $\lambda^* \cong \pi/L$ for $h \to 1$. For the case $h \to 0$, it is necessary to make the obvious replacements. The theorem is proved.

The existence of eigenvalues of the problem is proved only for a sufficiently large relative length of the profile in Theorem 2.1.

Theorem 2.2 (the existence of eigenoscillations). *There always exist eigenoscillations near a profile in a channel, irrespective of the length and position of the profile.*

Proof. It suffices to show that, for any value of $L > 0$, there exists $R > 0$ such that the inequalities

$$0 < \lambda_{\mathrm{NR}} \leqslant \lambda^* \leqslant \lambda_{\mathrm{DR}} < \pi \tag{2.9}$$

hold.

Estimate from Below. If $R > L/2$, we have $\lambda_{\mathrm{NR}} > 0$ by virtue of the connected region and the orthogonality of the solution to the constant, because $(\lambda_{\mathrm{NR}})^2$ is the second eigenvalue of the Neumann problem for the Laplace operator in a bounded connected region.

Estimate from Above. Let the continuous component u_c of the approximate eigenfunction u in Ω be of the form $u_c = \cos(\pi\zeta)\cos(\pi\xi/R)$ in the representation (2.2).

The discontinuous-at-the-profile component of the approximate eigenfunction (2.3) has the form

$$u_{\rm d}^{*} = \begin{cases} æ\cos(\pi\xi/L), & (\xi,\zeta) \in 1, \\ (1-1/h)\,æ\cos(\pi\xi/L), & (\xi,\zeta) \in 2, \end{cases}$$

where $æ$ is an arbitrary constant. The function $u_{\rm d}$ can be regarded as a function in the entire region of oscillations if it is extended identically equal to zero outside regions 1 and 2. For all the values of $æ$, the relation

$$(\lambda_{\rm DR})^2 \leqslant \int\limits_{\Omega_R} [\nabla(u_{\rm c}+u_{\rm d})]^2\, d\Omega_R \Big/ \int\limits_{\Omega_R} (u_{\rm c}+u_{\rm d})^2\, d\Omega_R = \mu^2(æ, R), \tag{2.10}$$

which reflects the variational property of eigenvalues, holds, where $\Omega_R = \Omega \cap \{(\xi,\zeta) : |\xi| \leqslant R\}$. Direct calculation is used to check whether the asymptotic representation

$$\mu^2(æ,R) \cong \pi^2 + \frac{A}{R} + \frac{B}{R^2}$$

holds for large values of R (this is because of the boundedness of the support of the function $u_{\rm d}$). The quantities A and B are $æ$-dependent. Since the parameters R and $æ$ are independent, the A value is determining for sufficiently large R. The following relation holds:

$$A = \frac{(L^2h - L^2 + 1 - h)\pi^2 æ^2}{hL} + 8\,\frac{\sin(\pi h)Læ}{h}.$$

It follows that the values of A will be negative for sufficiently small negative values of $æ$. In view of this, the rigorous inequality $\mu^2(æ,R) < \pi^2$ holds for sufficiently large values of R and small negative values of $æ$.

Inequalities (2.9) hold by virtue of relation (2.10). The theorem is proved.

Remark 2.3. The method of proving Theorem 2.2 is, as a matter of fact, based on the estimate of the profile-introduced perturbation of the eigenfunction. It follows that as the profile length decreases, the eigenfunction increasingly "resembles" the generalized eigenfunction subject to conditions (1.5).

Remark 2.4. The mechanics of eigenoscillations for $L > 1$ and $L < 1$ has no principal differences. For $L < 1$, the eigenfunctions can be localized between the profile and the channel walls. This fact is of significance for $h \to 0$ or $h \to 1$. In this case, the eigenfunction is "squeezed" outside from the space between the profile and the wall (the smallness of the quantity $æ$). If $L > 1$, it follows from the proof of Theorem 2.1 that the eigenfunction is localized between the profile and the nearest channel wall for $h \to 0$ or $h \to 1$. Note that the behavior of the smallest eigenvalues is discussed, unless otherwise specified.

3. DISCRETIZATION OF THE PROBLEM AND NUMERICAL STUDIES

Discretization of the problem should take into account all its properties. The most specific task, which is typical of problems of this kind alone, is making allowance for the energy-finiteness condition. If relations (2.7) are regarded as equalities of the Fourier series on the entire interval $g_{(1-3)}$ and $g_{(2-3)}$ with respect to the variable ζ, they take the form of an infinite homogeneous system of equations [7].

The matrix elements of these equations depend analytically on the parameter λ. This system has a serious disadvantage for the numerical methods of solution on which we have to focus our attention. Additional conditions are needed [7] to convert the numerical solution of these equations to a solution for which the energy-finiteness conditions hold. In the present study, for correctness of the calculations for $h = 1/2$, the discretized relations (2.7) were supplemented by a forced energy-finiteness condition, which allows us to increase significantly the accuracy and speed of calculation. By virtue of Corollary 2.1, the equality

$$(a_0 - b_0)\cos\left(\lambda\frac{L}{2}\right) + \sum_{m=1}^{\infty}(a_m - b_m)\cosh\left\{\frac{L}{2}\sqrt{\left[\frac{m\pi}{1-h}\right]^2 - \lambda^2}\right\} = 0 \tag{3.1a}$$

holds at the profile edges whose oscillation modes are even along the abscissa, and the equality

$$(a_0 - b_0)\sin\left(\lambda\frac{L}{2}\right) + \sum_{m=1}^{\infty}(a_m - b_m)\sinh\left\{\frac{L}{2}\sqrt{\left[\frac{m\pi}{1-h}\right]^2 - \lambda^2}\right\} = 0 \tag{3.1b}$$

holds for odd modes.

Relations (3.1a) and (3.1b) have the sense of a forced satisfaction of the energy-finiteness condition at the profile edges for approximate eigenfunctions, and they are supplementary in eigenvalue and eigenfunction calculations (the number of unknown $\{a_m\}$ and $\{b_m\}$ is equal). This makes it possible to use the results of [7] to support numerical studies. To determine numerically the dependence of the eigenvalues and the eigenfunctions on the parameter h, use was made of the generalized expansion method for the determinant. The known technique for the solution of infinite systems of equations in special spaces of permissible solutions [7] was employed to satisfy the energy-finiteness condition in discrete models.

Method of Direct Forced Allowance for the Energy Finiteness. This method is used here for a numerical study of the dependence of the frequency of eigenoscillations on the profile length in the case where the profile is in the center of a channel [$h = (1-h) = 1/2$]. Therefore, conditions (2.6) take the form $a_0 = -b_0$. If the profile is in the center of the channel, any eigenfunction u^* of the OEO problem can be represented as a sum of two functions $u^* = u_a + u_s$ where the function u_s is symmetrical in the profile and u_a is antisymmetrical. Since any eigenfunction of the OEO problem is subject to condition (1.5), one can consider that the eigenfunction u^* is antisymmetrical relative to the profile if it is in the center of the channel. To do this, one can consider that $u_1 = -u_2$ or $a_m = -b_m$ for all values of m. In addition, since the function u_3 is antisymmetrical relative to the location of the profile, all the coefficients c_k with even numbers are equal to zero in representations (2.4) and (2.5). The system of equations derived from (2.7) with condition (3.1) was separated and studied numerically.

Expansion Method for the Determinant. The greatest difficulty encountered in numerical analysis of an OEO-type problem lies in the fact that the convergence of approximate solutions to the solution of the problem in the class of functions with energy-finiteness conditions at the edge should be controlled.

Let the eigenfunctions be approximately representable in the following form for regions 1 (over the profile) and 2 (under the profile):

$$u_1(\xi,\zeta) = a_0\left\{\begin{array}{c}\cos(\lambda\xi)\\ \sin(\lambda\xi)\end{array}\right\} + \sum_{m=1}^{M} a_m\cos\left[\frac{m\pi(\zeta-h)}{1-h}\right]\left\{\begin{array}{c}\cosh(\xi\alpha_m)\\ \sinh(\xi\alpha_m)\end{array}\right\},$$

$$u_2(\xi,\zeta) = b_0\left\{\begin{array}{c}\cos(\lambda\xi)\\ \sin(\lambda\xi)\end{array}\right\} + \sum_{n=1}^{N} b_n\cos\left[\frac{n\pi(\zeta-h)}{h}\right]\left\{\begin{array}{c}\cosh(\xi\beta_n)\\ \sinh(\xi\beta_n)\end{array}\right\},$$

$$\alpha_m = \sqrt{\left(\frac{m\pi}{1-h}\right)^2 - \lambda^2}, \qquad \beta_n = \sqrt{\left(\frac{n\pi}{h}\right)^2 - \lambda^2}.$$

These relations depend analytically on λ and contain $K = N + M + 1$ of unknown constants a_m and b_n, because a_0 and b_0 are related by condition (3.1). To determine these unknowns, we need K additional relations, which can be derived using conditions (2.7). As a matter of fact, K partial radiation conditions should be satisfied for an approximate eigenfunction. Let

$$u_3(\xi,\zeta) = \sum_{k=1}^{K+1} c_k\cos(k\pi\zeta)\exp(-\xi\delta_k), \quad \delta_k = \sqrt{(k\pi)^2 - \lambda^2}.$$

Conditions (2.7) lead to the following system of relations for the unknown coefficients c_k ($k = 1,\ldots,K = M + N + 1$):

$$c_k = \int_h^1 u_1\cos(k\pi\zeta)\,d\zeta + \int_0^h u_2\cos(k\pi\zeta)\,d\zeta,$$

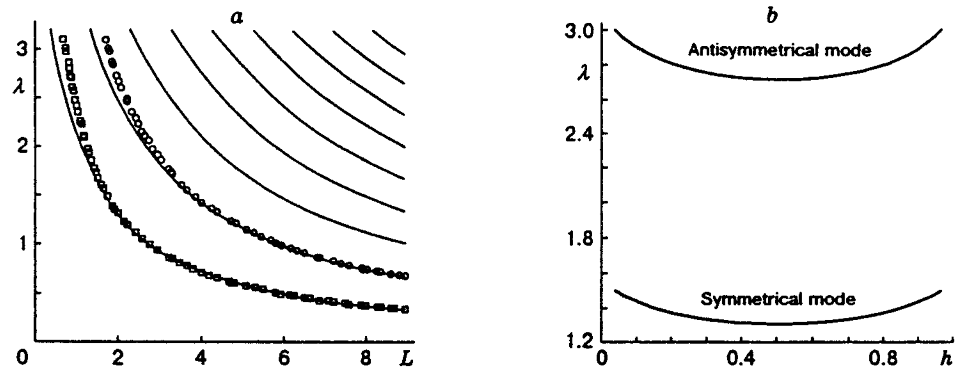

Fig. 2. Frequencies of eigenoscillations versus the profile length (a) and its position in the channel (b): (a) solid curves refer to the results obtained by the method of direct forced allowance for the finite energy and points refer to the results obtained using relation (3.2).

$$c_k = -\frac{1}{\delta_k}\Big\{\int\limits_0^h \Big(\frac{d}{d\xi}u_2\Big)_{\xi=L/2}\cos(k\pi\zeta)\,d\zeta + \int\limits_h^1 \Big(\frac{d}{d\xi}u_1\Big)_{\xi=L/2}\cos(k\pi\zeta)\,d\zeta\Big\}.$$

If c_k is excluded from these relations, we derive the homogeneous system of equations with K unknown constants a_m and b_n ($m = 0, 1, \ldots, M$ and $n = 1, \ldots, N$):

$$\int\limits_0^h \cos(k\pi\zeta)\Big(u_2 + \frac{1}{\delta_k}\frac{d}{d\xi}u_2\Big)_{\xi=L/2} d\zeta + \int\limits_h^1 \cos(k\pi\zeta)\Big(u_1 + \frac{1}{\delta_k}\frac{d}{d\xi}u_1\Big)_{\xi=L/2} d\zeta = 0 \quad (k = 1, \ldots, K).$$

After the variables are replaced, the system takes the canonical form

$$a_0\Big(\frac{1}{\delta_k - i\lambda} + \frac{e^{-i\lambda L}}{\delta_k + i\lambda}\Big) + \sum_{m=1}^{M} a_m\Big(\frac{1}{\delta_k + \alpha_m} + \frac{e^{-\alpha_m L}}{\delta_k - \alpha_m}\Big) + \sum_{n=1}^{N} b_n\Big(\frac{1}{\delta_k + \beta_n} + \frac{e^{-\beta_n L}}{\delta_k - \beta_n}\Big) = 0.$$

It was studied using the expansion method for the determinant [7]. The relation that permits one to calculate approximately the eigenfrequencies of oscillations near the profile in the channel was obtained:

$$k\pi = \lambda\Big(L - 2\frac{(1-h)\ln(1-h) + h\ln(h)}{\pi}\Big) - \arctan\Big(\frac{2\lambda}{\sqrt{\pi^2/(1-h)^2 - \lambda^2}}\Big)$$
$$- \arctan\Big(\frac{2\lambda}{\sqrt{\pi^2/h^2 - \lambda^2}}\Big) + \arctan\Big(\frac{2\lambda}{\sqrt{\pi^2 - \lambda^2}}\Big) \qquad (3.2)$$

(k is an odd number for eigenoscillations that are even along the ξ axis and k is an even number for odd oscillations).

4. NUMERICAL STUDIES

Dependence of the Frequency on the Length of a Profile and Its Position in a Channel. Figure 2a shows results of numerical studies of the frequencies of eigenoscillations versus the length of the plate profile. Even and odd modes alternate. Comparison of the numerical results clearly shows that as the plate lengthens, the values of the frequencies of the first modes of eigenoscillations decrease, and the modes increase in number. The largest possible frequency for each mode is the π value attainable as the profile length is decreased to some critical value at which the modes of eigenoscillations change in number. For the first mode, we have $\lim\limits_{L\to 0}[\lambda_1(L)] = \pi$. Figure 2b shows calculated data on the dependence of the frequency of

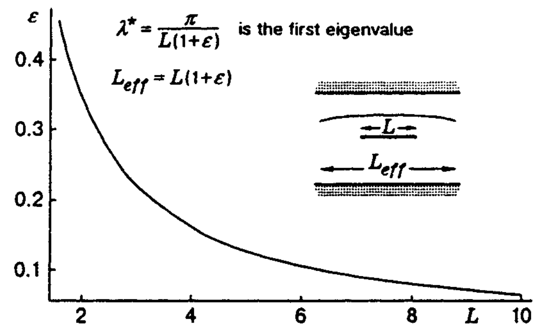

Fig. 3. Effective length (for the first eigenfrequency) versus the profile length.

eigenoscillations on the position of the profile for its fixed length. One should focus attention on the weak dependence of the frequencies on h. For $h \to 0$ or $h \to 1$, in the case where $L > 1$, the ultimate transition $\lambda(h) \to \pi/L$ is observed. If $L < 1$, the ultimate transition $\lambda(h) \to \pi$ holds. In the latter case, half the wavelength of eigenoscillations tends to the channel height.

Eigenoscillations exist for any lengths and any position of the plate in the channel, and, for any fixed length of the plate profile, the number of the modes of eigenoscillations is finite and is determined from Fig. 2a.

Effective Length of a Profile. Let $L \gg 1$. If the profile length tends to infinity or the profile approaches the channel wall, the first dimensionless frequency of eigenoscillations near the profile λ^* has approximately the form $\lambda^* \approx \pi/L$. By virtue of this, to study the mechanics of oscillations at the first eigenfrequency, it is convenient to introduce the notion of an effective profile length L_{eff}, which equals half the wavelength of eigenoscillations ($L_{\text{eff}} = \pi/\lambda^*$). By virtue of Theorems 2.1 and 2.2, the following propositions hold.

Proposition 4.1 (the dependence of the effective length on the limiting length of a profile). *If $L \to 0$, we have $L_{\text{eff}} \to 1$. If $L \to \infty$, we have $L_{\text{eff}} \to L$.*

Proposition 4.2 (the dependence of the effective length on the position of a profile). *Let $L > 1$. If $h \to 0$ ($h \to 1$), we have $L_{\text{eff}} \to L$. Let $L < 1$. If $h \to 0$ ($h \to 1$), we have $L_{\text{eff}} \to 1$.*

Let $L_{\text{eff}} = \pi/\lambda^* = L(1+\varepsilon)$ or $L_{\text{eff}}/L - 1 = \varepsilon$. Figure 3 illustrates the quantity ε versus the dimensionless profile length, which was obtained numerically by the method of forced allowance for energy finiteness. One can note the good agreement between the calculation results and the above propositions.

Theoretical and experimental studies allows us to conclude that there is no minimum or maximum profile length at which eigenoscillations are absent.

Form of Eigenfunctions and Dependence of the Amplitude on the Coordinates. The dependences of the eigenfunction on the coordinates for $L = 2$ and $h = 1/2$ were calculated using the method of forced allowance for energy finiteness. Figure 4a shows the amplitude under the plate for the first (even) mode of eigenoscillations versus the coordinates. Owing to the orthogonality of the eigenoscillations to the piston mode, the eigenfunction is antisymmetrical relative to the profile if it is in the center of the channel (the oscillations above and below the profile are in antiphase).

Form of Eigenoscillations and Direction of Acoustic Flow Velocities. Figure 4b shows the field of acoustic velocities for the first mode of eigenoscillations (the compression phase above the plate and the rarefaction phase below the plate). The studies allow us to clarify the mechanics of eigenoscillations near the profile in the channel. Figure 4b demonstrates distinctly that the first mode of eigenoscillations is the gas flow from region 1 to region 2 and backward.

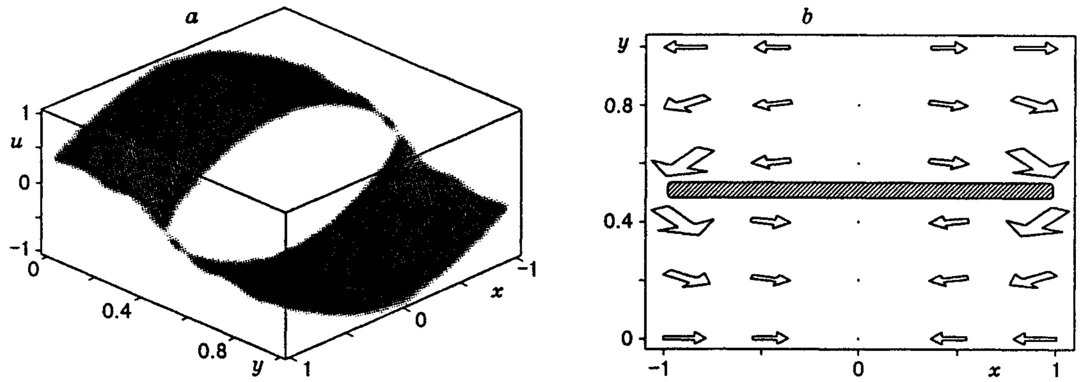

Fig. 4. Oscillation amplitude (a) in the velocity field (b) of the first mode in the neighborhood of the profile.

CONCLUSIONS

• A mathematical model that describes eigenoscillations near a plate in a channel has been constructed. The eigenoscillations have been completely studied numerically.

• The dependences of the frequencies of eigenoscillations on the profile length and its position in the channel and the effective profile length have been studied numerically.

• Eigenoscillations have been shown to exist for any lengths and positions of the profile in the channel.

• The asymptotics of the eigenfrequencies as the plate approaches the channel wall and the profile length lengthens infinitely or shortens have been studied. In approaching the plate to the channel wall, the eigenfunction has been shown to be localized between the plate and the wall if the profile length is larger than the channel height. If its length tends to zero, the eigenfrequency of oscillations tends to the lowest frequency which corresponds to the admissible generalized eigenwaves in an empty channel.

• The amplitude of eigenoscillations versus the coordinates has been examined. The eigenoscillations above and below the profile have been shown to be in antiphase.

The author is grateful to R. M. Garipov for a helpful discussion of the work.

This work was supported by the Russian Foundation for Fundamental Research (Grant No. 95-01-00894).

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